

Conformal Gravity with the Dirac Fields

Luca Fabbri

Abstract

In a recent paper we have constructed the conformal theory of gravitation with both torsion and metric; this background having both torsion and metric is the best adapted to host matter field having both spin and energy and therefore in this paper we shall study the simplest spinor field described by the Dirac field deriving the matter field equations together with the spin and energy densities and eventually we check that both conservation laws and trace condition are satisfied.

Introduction

That conformal gravitation is important is due to several reasons: their lagrangian is unique, their renormalizability for the problem of quantization, their dynamics appear to fit the curves of rotation of galaxies for the issue of dark matter; moreover there exist mechanisms of gravitational spontaneous conformal symmetry breaking, inducing also gauge symmetry breaking [1, 5]. On the other hand however, all these results are given for the metric case neglecting torsion; however metric as well as torsional conformal transformations can be defined [6]. The impasse is met whenever one tries to define a conformally covariant curvature with both metric and torsional degrees of freedom.

In a very recent paper we have found such a conformally covariant curvature in $(1 + 3)$ -dimensional spacetimes with metric and torsion; then we have constructed the conformal gravitational metric-torsional theory in terms of the field equations for the energy and spin densities of generic matter fields [7].

In this paper, we study the specific example of Dirac matter field with both spin and energy density in gravitational conformal theories.

1 Conformal Curvature and Derivative

In this paper we shall follow all notation and conventions of [7]. In particular the metric tensor $g_{\alpha\beta}$ and the Cartan torsion tensor $Q_{\sigma\rho\alpha}$ will be used to build the metric-compatible connection as

$$\Gamma_{\rho\alpha}^{\sigma} = \frac{1}{2}g^{\sigma\theta}[Q_{\rho\alpha\theta} + Q_{\alpha\rho\theta} + Q_{\theta\rho\alpha} + (\partial_{\rho}g_{\alpha\theta} + \partial_{\alpha}g_{\rho\theta} - \partial_{\theta}g_{\rho\alpha})] \quad (1)$$

while the constant Minkowskian metric η_{ij} and a basis of vierbein e_{α}^i such that we have the relationship $e_{\alpha}^p e_{\nu}^i \eta_{pi} = g_{\alpha\nu}$ will be used to pass from such a metric-compatible connection to the antisymmetric spin-connection

$$\omega_{\alpha}^{ij} = \eta^{pj}[e_{\sigma}^i(\Gamma_{\rho\alpha}^{\sigma} e_p^{\rho} + \partial_{\alpha}e_p^{\sigma})] = -\omega_{\alpha}^{ji} \quad (2)$$

defining two formalisms that are equivalent; nevertheless despite their equivalence, these formalisms respectively called spacetime and world formalism are different for what regards their employment: in fact, in the spacetime formalism there is no distinction between the local and structural information, both contained in the metric tensor, whereas in the world formalism there is a separation between local and structural information, since the former are contained in the vierbein while the latter are given by the Minkowskian metric. Indeed, we have that all local information must be contained within the vierbein, because it can not be contained within the constant Minkowskian metric; on the other hand the Minkowskian structure is given in terms of the Minkowskian metric by definition: moreover, we have that the covariant constancy of the metric defines the connection (1) as the covariant constancy of the vierbein defines the relationship between the connection and the spin-connection (2), but it is the covariant and ordinary constancy of the Minkowskian metric that determines the fact that the spin-connection is antisymmetric. This point is fundamental due to the fact that the Lorentz structure is represented for the spin-connection by its antisymmetry. To better explain this point, we recall that the Lorentz group admits also a complex representation; the complex representation is given in terms of the complex γ_a matrices verifying the relations $\{\gamma_a, \gamma_b\} = 2\mathbb{I}\eta_{ab}$ known as the Clifford algebra, from which it is further possible to define the linearly independent complex σ_{ab} matrices given by $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ as the complex generators of the complex representation of the Lorentz algebra: now it is possible to see that with both the antisymmetric complex generators and the antisymmetric spin-connection we build the complex spinor-connection given by the following expression

$$\Omega_\rho = \frac{1}{2}\omega_\rho^{ij}\sigma_{ij} \quad (3)$$

as the complex spinor-connection defining a complex spinorial covariant derivatives with respect to which we have the constancy of the γ_a matrices. Finally by exponentiating the complex generators with respect to the local parameters of the Lorentz transformation one obtains the complex matrix \mathbf{S} describing the complex Lorentz group: eventually we have that the complex spinor field denoted with ψ is defined to undergo $\psi' = \mathbf{S}\psi$ as transformation law.

Now the conformal transformation is given as in [7]. In particular given the function σ we have that the conformal transformation for the metric it is expressed by

$$g_{\alpha\beta} \rightarrow \sigma^2 g_{\alpha\beta} \quad (4)$$

while by defining $\ln \sigma = \phi$ we have that for the torsion tensor it is given by the following

$$Q^\sigma_{\rho\alpha} \rightarrow Q^\sigma_{\rho\alpha} + q(\delta^\sigma_\rho \partial_\alpha \phi - \delta^\sigma_\alpha \partial_\rho \phi) \quad (5)$$

in terms of the parameter q as the most general possible, and from the relationship (1) it is possible to see what is the conformal transformation for the connection in its most general form; the conformal transformation for the vierbein is therefore

$$e^k_\alpha \rightarrow \sigma e^k_\alpha \quad (6)$$

and by employing the relationship (2) it is possible to get the conformal transformation for the spin-connection. Finally we have that because there is no conformal transformation for the constant γ_a matrices then from the relationship (3) we obtain the conformal transformation for the spinor-connection: the conformal transformation of spinor fields

$$\psi \rightarrow \sigma^{-\frac{3}{2}} \psi \quad (7)$$

is chosen for the specific case given by the Dirac spinor field.

In this framework, the Riemann-Cartan metric-torsional curvature tensor is defined as in

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2} G^i_{k\mu\nu} \eta^{kj} \sigma_{ij} = \frac{1}{2} [G^\rho_{\xi\mu\nu} e^i_\rho e^\xi_k] \eta^{kj} \sigma_{ij} = \\ &= \frac{1}{2} [(\partial_\mu \Gamma^\rho_{\xi\nu} - \partial_\nu \Gamma^\rho_{\xi\mu} + \Gamma^\rho_{\sigma\mu} \Gamma^\sigma_{\xi\nu} - \Gamma^\rho_{\sigma\nu} \Gamma^\sigma_{\xi\mu}) e^i_\rho e^\xi_k] \eta^{kj} \sigma_{ij} \equiv \\ &\equiv \frac{1}{2} [\partial_\mu \omega^i_{k\nu} - \partial_\nu \omega^i_{k\mu} + \omega^i_{a\mu} \omega^a_{k\nu} - \omega^i_{a\nu} \omega^a_{k\mu}] \eta^{kj} \sigma_{ij} \equiv \\ &\equiv \partial_\mu \Omega_\nu - \partial_\nu \Omega_\mu + \Omega_\mu \Omega_\nu - \Omega_\nu \Omega_\mu \end{aligned} \quad (8)$$

in which there is the implicit presence of the Cartan torsion tensor. The commutator of complex spinorial covariant derivatives can be expressed as in the following expression

$$[D_\rho, D_\mu] \psi = Q^\theta_{\rho\mu} D_\theta \psi + G_{\rho\mu} \psi \quad (9)$$

in which there is an explicit Cartan torsion and an implicit Cartan torsion tensor contained with the explicit Riemann-Cartan curvature tensor itself.

At this point it is possible to construct the metric-torsional curvature tensor

$$M_{\alpha\beta\mu\nu} = G_{\alpha\beta\mu\nu} + (\frac{1-q}{3q})(Q_\beta Q_{\alpha\mu\nu} - Q_\alpha Q_{\beta\mu\nu}) \quad (10)$$

from which we construct the modified metric-torsional curvature tensor as in the following

$$T_{\alpha\beta\mu\nu} = M_{\alpha\beta\mu\nu} - \frac{1}{2}(M_{\alpha[\mu} g_{\nu]\beta} - M_{\beta[\mu} g_{\nu]\alpha}) + \frac{1}{12} M(g_{\alpha[\mu} g_{\nu]\beta} - g_{\beta[\mu} g_{\nu]\alpha}) \quad (11)$$

for any value of the parameter q and in which there is implicit and explicit torsion; these two are antisymmetric in both the first and second couple of indices but the latter is irreducible and also conformally covariant. So we have constructed the metric-torsional conformal curvature in $(1+3)$ -dimensional space-times. As for the form of the spinorial covariant derivative of the spinor field in this specific case there is no need of any improved definition to achieve the conformal covariance. Thus we maintain the dynamical term considered as usual.

2 Conformal Gravity and Matter: the case of the Dirac Fields

As in [7] we have constructed the conformal gravitational matter field in general we shall now focus on the specific example given by the simplest Dirac field.

That the the spinorial covariant derivative for the spinor field in this specific case does not need an improved definition to achieve conformal invariance is

clear from the fact that the Dirac action is already able to accomplish conformal invariance and so we shall simply maintain the form

$$S = \int [L_{\text{gravity}} + \frac{i}{2}(\bar{\psi}\gamma^\mu \mathbf{D}_\mu \psi - \mathbf{D}_\mu \bar{\psi}\gamma^\mu \psi)] |e| dV \quad (12)$$

with constant k and where it is over the volume of the spacetime that the integral is taken. By varying this action with respect to the spinor field one obtains the matter field equations as

$$i\gamma^\mu \mathbf{D}_\mu \psi + \frac{i}{2} Q_\mu \gamma^\mu \psi = 0 \quad (13)$$

while by varying it with respect to the metric and connection or equivalently vierbein and spin-connection one obtains the completely antisymmetric irreducible spin and energy densities as

$$S_{\mu\alpha\beta} = \frac{1}{4} \varepsilon_{\mu\alpha\beta\rho} \bar{\psi} \gamma^\rho \gamma^\mu \psi \quad (14)$$

and

$$T_{\mu\alpha} = \frac{i}{2} (\bar{\psi} \gamma_\mu \mathbf{D}_\alpha \psi - \mathbf{D}_\alpha \bar{\psi} \gamma_\mu \psi) \quad (15)$$

in the system of field equations for gravitational dynamics that are found in [7].

By taking into account the definition of the commutator of spinorial covariant derivatives we have that once the conformal matter field equations (13) are considered then the completely antisymmetric spin and energy densities (14-15) satisfy the conservation law given by the trace condition

$$(1 - q)(D_\mu S_\nu{}^{\nu\mu} + Q_\mu S_\nu{}^{\nu\mu}) + \frac{1}{2} T_\mu{}^\mu = 0 \quad (16)$$

in a trivial way and the other two conservation laws

$$D_\rho S^{\rho\mu\nu} + Q_\rho S^{\rho\mu\nu} + \frac{1}{2} T^{[\mu\nu]} = 0 \quad (17)$$

and

$$D_\mu T^{\mu\rho} + Q_\mu T^{\mu\rho} - T_{\mu\sigma} Q^{\sigma\mu\rho} + S_{\beta\mu\sigma} G^{\sigma\mu\beta\rho} = 0 \quad (18)$$

that is in the present situation the complete antisymmetry of the spin density implies the irreducibility of the spin density and because of the masslessness then also the energy density is irreducible; therefore we have here the same traceless conditions we would have had in the standard case. We know that in general when a geometrical background is conformally invariant there is the removal of one degree of freedom compensated by the introduction of the conservation law of the trace as a constraint although in the present case we have that the tracelessness of the conserved quantities makes the conservation law of the trace trivial, and because the massless Dirac field is already conformal that the conservation law for the trace provides no additional information is expected.

3 Conformal Gravity with the Dirac Fields. Summary and Comments

Now that we have obtained the matter field equations we can join the results previously obtained in order to display the entire system of field equations as

$$i\gamma^\mu \mathbf{D}_\mu \psi + \frac{i}{2} Q_\mu \gamma^\mu \psi = 0 \quad (19)$$

along with

$$\begin{aligned}
& 4k[D_\rho P^{\alpha\beta\mu\rho} + Q_\rho P^{\alpha\beta\mu\rho} - \frac{1}{2}Q^\mu_{\rho\theta} P^{\alpha\beta\rho\theta} - \\
& - (\frac{1-q}{3q})2Q_\rho P^{\rho\alpha\beta\mu}] \varepsilon_{\alpha\beta\mu\sigma} = \\
& = -\frac{3}{2}\bar{\psi}\gamma_\sigma\gamma\psi
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
& 2k[P^{\theta\sigma\rho\alpha}T_{\theta\sigma\rho}{}^\mu - \frac{1}{4}g^{\alpha\mu}P^{\theta\sigma\rho\beta}T_{\theta\sigma\rho\beta} + P^{\mu\sigma\alpha\rho}M_{\sigma\rho} + \\
& + (\frac{1-q}{3q})(D_\nu(2P^{\mu\rho\alpha\nu}Q_\rho) + Q_\nu(2P^{\mu\rho\alpha\nu}Q_\rho - P^{\mu\nu\rho\sigma}Q^\alpha_{\rho\sigma}))] = \\
& = \frac{i}{4}(\bar{\psi}\gamma^\alpha\mathbf{D}^\mu\psi - \mathbf{D}^\mu\bar{\psi}\gamma^\alpha\psi)
\end{aligned} \tag{21}$$

for the Dirac matter fields with completely antisymmetric spin and energy densities in gravitational conformal theories independently on the value that the parameter q may have; thus there is a set of infinite conformal theories corresponding to the infinite values that the parameter q can assume: consequently this situation will provide the possibility for which some special fine-tuning for the model can be chosen in terms of specific values for the parameter q according to any reason that may eventually appear to be necessary.

Finally we remark that the complete antisymmetry of the spin density implies the relationships expressed by

$$\begin{aligned}
& D_\rho(P^{\alpha\beta\mu\rho} + P^{\alpha\mu\beta\rho}) + Q_\rho(P^{\alpha\beta\mu\rho} + P^{\alpha\mu\beta\rho}) - \\
& - \frac{1}{2}(Q^\mu_{\rho\theta}P^{\alpha\beta\rho\theta} + Q^\beta_{\rho\theta}P^{\alpha\mu\rho\theta}) - (\frac{1-q}{3q})Q_\rho(P^{\rho\beta\mu\alpha} + P^{\rho\mu\beta\alpha}) = 0
\end{aligned} \tag{22}$$

which are 20 constraints imposed on the geometry, although in the present case these constraints are differential constraints on the dynamics of the conformal curvature and torsion tensors and not accounted by requiring the complete antisymmetry of the torsion tensor as in the standard case; as the conformal transformation law for torsion conformally transforms only the torsion trace vector it is clear that it is not a conformally meaningful procedure to demand the vanishing of the torsion trace vector: therefore it is a property of conformal geometries that we have no complete antisymmetry for torsion. On the other hand, one may ask whether relationships (22) may eventually be satisfied by a set of 20 constraints given in a simpler form; this simpler form may perhaps consist in the vanishing of some tensor or some of its irreducible decompositions, but to this purpose we have to keep in mind that we can only require the vanishing of conformally invariant tensors and their conformally invariant irreducible decompositions: we know that the torsion tensor is decomposable in torsion trace vector, completely antisymmetric dual of an axial vector and the remaining traceless part, and the conformal transformation acts on the former alone, leaving unchanged the latter two, so that it makes sense to demand that the torsion tensor coincides with its trace vector, forcing the vanishing of the other irreducible parts. According to this requirement the 20 constraints would be accounted precisely, and in addition we would have that the tensorial conformal curvature defined here would reduce to the Weyl curvature defined in the metric case so that we would have $T^{\alpha\beta\mu\rho} \equiv W^{\alpha\beta\mu\rho}$ identically; then relationships (22) are satisfied if the fine-tuning $2A + 2B + C = 0$ were given. The consequences of this requirement complemented with such a fine-tuning is that when the tensorial conformal parametric curvature $P^{\alpha\beta\mu\rho}$ is decomposed in terms of the Weyl

curvature $W^{\alpha\beta\mu\rho}$ and Cartan torsion $Q^{\sigma\alpha\beta}$ in such a way that the contribution of the Riemann curvature $R^{\alpha\beta\mu\rho}$ would cancel leaving only the contributions of the Cartan torsion trace Q^μ losing all metric degrees of freedom leaving only the torsional ones, making the model degenerate. Thus if we decide to reduce the 20 constraints given by relationships (22) we find that the simplest way in which this could be achieved turns out to be meaningless, although this does not imply that reductions of more complicated structure could be accomplished.

Conclusion

We have considered the conformal gravity constructed in a previous paper, and in this paper we have discussed it in the case of the Dirac matter field equations with spin and energy densities showing that the matter field equations imply the spin and energy densities to satisfy traceless condition and conservation laws which entail the gravitational field equations to verify the Jacobi-Bianchi geometrical identities: we have discussed the fact that the conservation laws for the trace is trivial as a consequence that the massless Dirac field is already conformal invariant and we have considered the issue of the complete antisymmetry of the spin density of the Dirac field discussing some consequences.

References

- [1] K. S. Stelle, *Phys. Rev. D* **16**, 953 (1977).
- [2] P. D. Mannheim and D. Kazanas, *Astrophys. J.* **342**, 635 (1989).
- [3] A. Edery, L. Fabbri and M. B. Paranjape, *Class. Quant. Grav.* **23**, 6409 (2006).
- [4] A. Edery, L. Fabbri and M. B. Paranjape, *Can. J. Phys.* **87**, 251 (2009).
- [5] L. Fabbri, arXiv:0806.2610 [hep-th].
- [6] I. L. Shapiro, *Phys. Rept.* **357**, 113 (2002).
- [7] Luca Fabbri arXiv:1101.1761 [gr-qc].